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On almost regular tournament matrices

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Abstract

Spectral and determinantal properties of a special class $\mathscr{M}_n$ of $2n \times 2n$ almost regular tournament matrices are studied. In particular, the maximum Perron value of the matrices in this class is determined and shown to be achieved by the Brualdi–Li matrix, which has been conjectured to have the largest Perron value among all tournament matrices of even order. We also establish some determinantal inequalities for matrices in $\mathscr{M}_n$ and describe the structure of their associated walk spaces. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

A *tournament matrix* T is a (0, 1)-matrix that satisfies $T + T^{\mathrm{t}} = J - I$, where J is the all ones matrix and I is the identity matrix. Thus T has zero diagonal and $t_{ij} = 1$ if and only if $t_{ji} = 0$ for each $i \neq j$. The directed graph associated with a tournament matrix is known as a *tournament* and this is a well-studied class of graphs (see [16]). There is a growing body of work on tournament matrices (see references).

Given a tournament matrix T of order n, its *score vector* is given by Te where e is the all ones vector. The tournament matrix T is *regular* provided all of the entries in its score vector are the same, equivalently if each of the row sums of T is $(n-1)/2$. (Observe that necessarily n must be odd.) It is known that for odd n, the regular tournament matrices maximize the Perron value over the class of $n \times n$ tournament matrices [1]. If n is even, an $n \times n$ tournament matrix is called *almost regular* provided half of its row sums are $(n-2)/2$ and the other half are $n/2$. For even n, it is not known which $n \times n$ tournament matrices maximize the Perron value, but it is shown in [11] that for all sufficiently large n the maximizers are almost regular. In [3], Brualdi and Li conjecture that

$$B_n = \begin{bmatrix} U_{n/2} & U_{n/2}^{\mathrm{t}} \\ U_{n/2}^{\mathrm{t}} + I & U_{n/2} \end{bmatrix},$$

where U_k is the $k \times k$ strictly upper triangular matrix with ones above the main diagonal, maximizes the Perron value over the class of tournament matrices of even order n.

Inspired by this recent interest in almost regular tournament matrices we investigate some of their structural and algebraic properties in this paper. We focus particularly on the subclass $\mathscr{M}_n$ of almost regular tournament matrices of order $2n$ given by

$$\mathscr{M}_n = \left\{ \begin{bmatrix} T & T^{\mathrm{t}} \\ T^{\mathrm{t}} + I & T \end{bmatrix} : T \text{ is an } n \times n \text{ tournament matrix} \right\}.$$

Note that $\mathscr{M}_n$ contains the Brualdi–Li matrix B_{2n}. For a tournament matrix T, we let

$$M_T = \begin{bmatrix} T & T^{\mathrm{t}} \\ T^{\mathrm{t}} + I & T \end{bmatrix}.$$

The following result summarizes some of the spectral results from [1,12] on tournament matrices employed in this paper.

Proposition 1.1. *Let T be an $n \times n$ tournament matrix with eigenvalue λ and corresponding eigenvector v.*

(i) $-1/2 \leqslant \operatorname{Re} \lambda \leqslant (n-1)/2$;

(ii) $\operatorname{Re} \lambda = -1/2$ *if and only if* $e^{\mathrm{t}} v = 0$, *where e is the all ones vector. In this instance*, $T^{\mathrm{t}} v = \overline{\lambda} v$ *and v is orthogonal to any eigenvector corresponding to another eigenvalue of T*;

(iii) *the walk space W_T of T, defined by $\langle e, Te, \ldots, T^{n-1}e\rangle$, is the same as the subspace spanned by the eigenvectors and generalized eigenvectors of T corresponding to eigenvalues with real part strictly larger than $-1/2$;*
(iv) *there is an orthogonal basis of eigenvectors of T that spans $W_T^{\perp}$. Each eigenvector in $W_T^{\perp}$ corresponds to an eigenvalue whose real part is $-1/2$.*

Throughout the sequel we will refer to basic ideas and techniques from the theory of matrices. The reader is referred to [6,14] for further details.

2. Structure for almost regular tournament matrices

Our first result provides a construction for almost regular tournament matrices with certain specified submatrices.

Theorem 2.1. *Let S and T be tournament matrices of order n. There is an almost regular tournament matrix of order $2n$ of the form*

$$A = \begin{bmatrix} S & X \\ J - X^{\mathrm{t}} & T \end{bmatrix},$$

where the first n row sums of A are equal to $n-1$ and the last n row sums of A are equal to n.

Proof. Let the score vectors of S and T be s and t, respectively. Evidently, we need only prove the existence of an $n \times n$, $(0, 1)$-matrix X with row sum vector $(n-1)e - s$ and column sum vector t. According to, Corollary 6.2.3 of [4] such an X exists provided that for each pair of subsets α and β of $\{1, 2, \ldots, n\}$,

$$\sum_{i\in\alpha}(n-1-s_i) - \sum_{j\in\overline{\beta}} t_j \leqslant |\alpha|\,|\beta|\,.$$

Since every ℓ by ℓ principal submatrix of a tournament matrix has $\binom{\ell}{2}$1's,

$$\sum_{i\in\alpha} s_i \geqslant \binom{|\alpha|}{2} \qquad \text{and} \qquad \sum_{j\in\overline{\beta}} t_j \geqslant \binom{n-|\beta|}{2}.$$

Hence

$$\begin{aligned}\sum_{i\in\alpha}(n-1-s_i) - \sum_{j\in\overline{\beta}} t_j &\leqslant |\alpha|(n-1) - \binom{|\alpha|}{2} - \binom{n-|\beta|}{2} \\ &= |\alpha||\beta| - \frac{(|\alpha|-n+|\beta|)^2 + (|\alpha|-n+|\beta|)}{2}.\end{aligned}$$

Letting $\delta = |\alpha| - n + |\beta|$, which is an integer, we see $\delta^2 + \delta \geqslant 0$. Thus

$$\sum_{i\in\alpha}(n-1-s_i) - \sum_{j\in\overline{\beta}} t_j \leqslant |\alpha||\beta| - \frac{\delta^2+\delta}{2} \leqslant |\alpha||\beta|. \qquad \square$$

The next two results give alternative descriptions of the walk space of an almost regular tournament matrix. We will use the notation

$$u^{\mathrm{t}} = [\overbrace{1, 1, \ldots, 1}^{n} | \overbrace{0, 0, \ldots, 0}^{n}] \quad \text{and} \quad \ell^{\mathrm{t}} = [\overbrace{0, 0, \ldots, 0}^{n} | \overbrace{1, 1, \ldots, 1}^{n}]$$

in our discussion.

Theorem 2.2. *Let A be a $2n \times 2n$ almost regular tournament matrix such that the first n rows of A have sum $n-1$ and the last n rows have sum n. Then for each $j \geqslant 0$, $\langle e, \ell, A\ell, A^2 l, \ldots, A^j \ell \rangle = \langle e, Ae, A^2 e, \ldots, A^{j+1} e \rangle$. In particular, $W_A = \langle e, \ell, A\ell, A^2\ell, \ldots, A^{2n-2}\ell \rangle$.*

Proof. Note that the case $j = 0$ follows upon observing that $Ae = (n-1)\,e + \ell$. For $j \geqslant 1$, $A^{j+2}e = (n-1)\,A^{j+1}e + A^{j+1}\ell$, and hence $A^{j+2}e$ is a linear combination of $A^{j+1}e$ and $A^{j+1}\ell$, and $A^{j+1}\ell$ is a linear combination of $A^{j+2}e$ and $A^{j+1}e$. The result now follows by a simple inductive argument. □

Corollary 2.3. *If A is a $2n \times 2n$ almost regular tournament matrix with $n \geqslant 2$, then $W_A = \langle \ell, A\ell, A^2\ell, \ldots, A^{2n-1}\ell \rangle$.*

Proof. Let $R = \langle \ell, A\ell, A^2\ell, \ldots, A^{2n-1}\ell \rangle$. Then R is A-invariant by the Cayley–Hamilton theorem. Since W_A is A-invariant, it follows from Theorem 2.2 that $R \subseteq W_A$ and that $R = W_A$ if $e \in R$.

Suppose $R \neq W_A$. Since W_A is spanned by a set of (generalized) eigenvectors of A, there must be a (generalized) eigenvector v in W_A that is not in R. Among all such vectors, pick v to be of minimum height k. Let $w = (A - \lambda I)v$. Then either $w = 0$ or else w is a (generalized) eigenvector of height $k-1$. Thus $w \in R$.

By Theorem 2.2, $v = r + ae$ for some $r \in R$, and some nonzero scalar a. Premultiplying both sides of this equation by $A - \lambda I$ yields $w = Ar - \lambda r + aAe - a\lambda e$. Since $Ae = (n-1)e + \ell$, we have that $w - Ar + \lambda r - a\ell = a(n-1-\lambda)e$. Because $a \neq 0$, and $\lambda < n-1$, we are led to the contradiction that $e \in R$. Therefore $W_A = R$. □

3. The Brualdi–Li conjecture and the class $\mathscr{M}_n$

In this section, we prove that the Brualdi–Li matrix, B_{2n}, has the largest Perron value among the matrices in $\mathscr{M}_n$.

Theorem 3.1. *Suppose T is an $n \times n$ tournament matrix with $n \geqslant 2$ and let x be a real number with $0 < x < 2/(n-1)$. Then*

$$e_n^{\mathrm{t}}(I - xT)^{-1}e_n \geqslant \frac{(1+x)^n - 1}{x}$$

with equality if and only if T is permutationally similar to U_n.

Proof. Since the spectral radius of T is at most $(n-1)/2$, and $x < 2/(n-1)$, $(I - xT)^{-1}$ exists and is given by

$$(I - xT)^{-1} = \sum_{k=0}^{\infty} x^k T^k.$$

For $k \geqslant 0$, $e^{\mathrm{t}}T^k e$ equals the number of walks of length k in the tournament associated with T. If $k \leqslant n-1$, then since every tournament on $k+1$ vertices has a path of length k [18], we find that

$$e^{\mathrm{t}}T^k e \geqslant \binom{n}{k+1}.$$

It follows that

$$e^{\mathrm{t}}(I - xT)^{-1}e \geqslant \sum_{k=0}^{n-1} x^k \binom{n}{k+1}$$

with equality only if T has no walks of length n or more. Simple algebra and the binomial theorem yield

$$\sum_{k=0}^{n-1} x^k \binom{n}{k+1} = \frac{(x+1)^n - 1}{x}.$$

Thus, we have established the inequality. Up to isomorphism, the only tournament on n vertices with no walks of length n or more is the transitive one. Since the transitive tournament on n vertices has $\binom{n}{k+1}$ walks of length k for $k \leqslant n-1$, equality holds for this tournament. Hence equality holds if and only if T is permutationally similar to U_n. □

Theorem 3.2. *Let T be an n by n tournament matrix. Let ρ be the Perron value of M_T. Then*

$$2\rho^2 - 2(n-1)\rho - (n-1) \leqslant \left(\left(\frac{\rho+1}{\rho}\right)^{2n} + 1\right)^{-1}$$

and equality holds if and only if T is permutationally similar to U_n.

Proof. Let the Perron vector of M_T be conformally partitioned as $\begin{bmatrix} u \\ v \end{bmatrix}$, where u and v are $n \times 1$ vectors with $e^{\mathrm{t}}u + e^{\mathrm{t}}v = 1$. Then

$$Tu + T^{\mathrm{t}}v = \rho u \tag{3.1}$$

and

$$(T^{\mathrm{t}} + I)u + Tv = \rho v. \tag{3.2}$$

Adding these equations, using $e^{\mathrm{t}}u + e^{\mathrm{t}}v = 1$, and then solving for ρu yields

$$e - (\rho + 1)v = \rho u. \tag{3.3}$$

Pre-multiplying both sides of (3.3) by e^{t} gives

$$n - e^{\mathrm{t}}v = \rho. \tag{3.4}$$

Next multiply both sides of (3.3) by ρ and use (3.4) to get

$$\begin{aligned}\rho^2 v &= (J - T)\rho u + \rho T v\\ &= (\rho(\rho - n + 1))e - Te + (2\rho + 1)Tv.\end{aligned}$$

Thus,

$$(\rho^2 I - (2\rho + 1)T)v = \rho(\rho - n + 1)e - Te. \tag{3.5}$$

If $\rho^2 \leqslant (2\rho + 1)(n - 1)/2$, then $2\rho^2 - 2\rho(n - 1) - (n - 1) \leqslant 0$, and hence the desired inequality holds. Otherwise, $\rho^2 I - (2\rho + 1)T$ is a nonsingular M-matrix. Let w be the vector with $\left(\rho^2 I - (2\rho + 1)T\right) w = e$. By (3.5),

$$\begin{aligned}v &= \rho(\rho - n + 1)w - Tw\\ &= \rho(\rho - n + 1)w + \frac{1}{2\rho + 1}e - \frac{\rho^2}{2\rho + 1}w\\ &= \frac{\rho}{2\rho + 1}(2\rho^2 - 2\rho(n - 1) - (n - 1))w + \frac{1}{2\rho + 1}e.\end{aligned}$$

Pre-multiplying both sides of the above equation by e^{t} and using (3.4) implies that

$$n - \rho = \frac{\rho}{2\rho + 1}(2\rho^2 - 2\rho(n - 1) - (n - 1))e^{\mathrm{t}}w + \frac{n}{2\rho + 1},$$

which upon simplifying becomes

$$2n - 2\rho - 1 = (2\rho^2 - 2\rho(n - 1) - (n - 1))e^{\mathrm{t}}w. \tag{3.6}$$

By Theorem 3.1,

$$\begin{aligned}e^{\mathrm{t}}w &= e^{\mathrm{t}}(\rho^2 I - (2\rho + 1)T)^{-1}e\\ &= \frac{1}{\rho^2}e^{\mathrm{t}}\left(I - \frac{2\rho + 1}{\rho^2}T\right)^{-1} e\\ &\geqslant \frac{\left(\frac{\rho+1}{\rho}\right)^{2n} - 1}{2\rho + 1}.\end{aligned}$$

Now substitute in (3.6). Since $2\rho^2 - 2\rho(n - 1) - (n - 1) \geqslant 0$

$$(2\rho + 1)(2n - 2\rho - 1) \geqslant (2\rho^2 - 2\rho(n - 1) - (n - 1))(x - 1),$$

where $x = ((\rho + 1)/\rho)^{2n}$. Simple algebra now shows that

$$1 \geqslant (2\rho^2 - 2\rho(n - 1) - (n - 1))(x + 1)$$

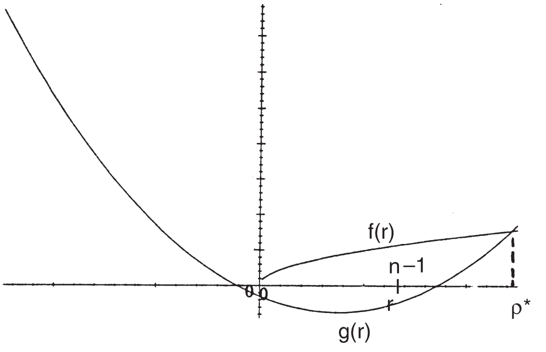

Fig. 1. Graphs of $f(r)$ and $g(r)$.

and the inequality follows. Note that equality implies equality in Theorem 3.1, and hence the T is permutationally similar to U_n. It is shown in [9], that if $T = U_n$, then equality holds. □

Theorem 3.3. *Let T be a tournament matrix of order n, let the Perron value of M_T be ρ, and let ρ^* denote the Perron value of the Brualdi–Li matrix B_{2n}. Then $\rho \leqslant \rho^*$, and equality holds if and only if M_T is permutationally similar to B_{2n}.*

Proof. From Theorem 3.2, we have

$$2\rho^2 - 2(n-1)\rho - (n-1) \leqslant \left(\left(\frac{\rho+1}{\rho}\right)^{2n} + 1\right)^{-1}.$$

If $n = 2$ or 3 the result is immediate, so suppose that $n \geqslant 4$. In [9] it is shown that

$$2\left(\rho^*\right)^2 - 2(n-1)\rho^* - (n-1) = \left(\left(\frac{\rho^*+1}{\rho^*}\right)^{2n} + 1\right)^{-1}.$$

It remains to show that $\rho \leqslant \rho^*$. To see this, we will prove that

$$\left(\left(\frac{r+1}{r}\right)^{2n} + 1\right)^{-1}$$

is concave down as a function of r for $r > n-1$, from which it will follow that for such r, if

$$2r^2 - 2(n-1)r - (n-1) \leqslant \left(\left(\frac{r+1}{r}\right)^{2n} + 1\right)^{-1}$$

then necessarily $r \leqslant \rho^*$. Let $f(r) = (((r+1)/r)^{2n} + 1)^{-1}$ and let $g(r) = 2r^2 - 2(n-1)r - (n-1)$. See Fig. 1. It can be shown that

$$f''(r) = \frac{2nr^{4n-2}(r+1)^{2n-2}}{\left[r^{2n} + (r+1)^{2n}\right]^3} \left[(2n - 2r - 1) \left(\frac{r+1}{r}\right)^{2n} - (2n + 1 + 2r) \right].$$

We are done if we can show

$$(2n - 2r - 1) \left(\frac{r+1}{r}\right)^{2n} < (2n + 1 + 2r) \quad \text{for } r > n - 1.$$

Observe that $(2n - 2r - 1)((r + 1)/r)^{2n}$ is decreasing in r, while $(2n + 1 + 2r)$ is increasing. Now

$$(2n - 2r - 1) \left(\frac{r+1}{r}\right)^{2n} < \left(1 + \frac{1}{n-1}\right)^{2n}.$$

Observe that $(1 + 1/(n - 1))^{2n}$ is decreasing in n. Since $n \geqslant 4$, we have $(1 + 1/(n - 1))^{2n} \leqslant (\frac{4}{3})^8 \doteq 9.9887$, while $2n + 1 + 2r \geqslant 4n - 1 \geqslant 15$. Thus we see that $f''(r) < 0$ for $r > n - 1$, so $f(r)$ is concave down. We then can conclude that $\rho \leqslant \rho^*$. Further if $\rho = \rho^*$, then

$$2\rho^2 - 2\,(n - 1)\,\rho - (n - 1) = \left(\left(\frac{\rho+1}{\rho}\right)^{2n} + 1\right)^{-1},$$

so by Theorem 3.2, T is transitive. It now follows that M_T is permutationally similar to B_{2n}. □

4. Other spectral properties for matrices in $\mathscr{M}_n$

In this section, we focus on the connection between the spectral properties of T and those of M_T.

Theorem 4.1. *Let T be a regular tournament matrix of order n. Then*
(i) *M_T is diagonalizable, and*
(ii) *if T has k distinct eigenvalues then M_T has $2k$ distinct eigenvalues.*

Proof. (i) Observe that since T is regular, the walk space of M_T has dimension 2. Thus by results in [8] M_T has a positive eigenvalue ρ, a negative eigenvalue r, and $2n - 2$ eigenvalues with real parts equal to $-1/2$. By Proposition 1.1 there is a collection of $2n - 2$ orthogonal eigenvectors of M_T corresponding to those eigenvalues with real part $-1/2$, it now follows that M_T is diagonalizable.

(ii) Since T is regular, it is normal and commutes with T^{t}. Hence the characteristic polynomial of M_T is given by $\det(zI - M_T) = \det\left(z^2 I - (2z + 1)\,T\right)$. Consequently z is an eigenvalue of M_T if and only if $z^2 = (2z + 1)\,\lambda$ for some eigenvalue λ of T. Suppose that z is an eigenvalue of M_T corresponding to two eigenvalues λ_1 and λ_2 of T. Then $z^2 = (2z + 1)\,\lambda_1$ and $z^2 = (2z+1)\,\lambda_2$, so that $(2z+1)\,(\lambda_1 - \lambda_2) = 0$.

Since the characteristic polynomial of M_T is a monic polynomial with integer coefficients, the eigenvalues are algebraic integers. Since $-1/2$ is not an algebraic integer, $z \neq -1/2$. So we deduce that $\lambda_1 = \lambda_2$. Note that Re $\lambda \geqslant -1/2$ and since T is regular, $\lambda \neq 0$, so $\lambda^2 + \lambda \neq 0$. Thus each distinct eigenvalue λ of T corresponds to two eigenvalues $\lambda \pm \sqrt{\lambda^2 + \lambda}$ of M_T. Thus T has k distinct eigenvalues and M_T has $2k$ distinct eigenvalues. $\square$

Theorem 4.2. *Suppose T is a tournament matrix of order n. Then w is an eigenvector of M_T corresponding to an eigenvalue γ with* Re $\gamma = -1/2$ *if and only if w has the form*

$$\begin{bmatrix} \sqrt{\overline{\lambda}}u \\ \pm\sqrt{-\lambda}u \end{bmatrix},$$

where u is an eigenvector of T corresponding to the eigenvalue λ with Re $\lambda = -1/2$. *When this is the case*, $\gamma = \lambda \pm \sqrt{-\lambda\overline{\overline{\lambda}}}$.

Proof. Suppose that $Tu = \lambda u$ and that Re $\lambda = -1/2$. Then $T^{\mathrm{t}}u = \overline{\lambda}u$ and it is readily verified that

$$\begin{bmatrix} \sqrt{\overline{\overline{\lambda}}}u \\ \pm\sqrt{-\lambda}u \end{bmatrix}$$

is an eigenvector of M_T corresponding to the eigenvalue $\lambda \pm \sqrt{-\lambda\overline{\overline{\lambda}}}$.

Conversely, suppose that w is an eigenvector of M_T corresponding to an eigenvalue γ with Re $\gamma = -1/2$, and partition w conformally with M_T as

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Then $e^{\mathrm{t}}w = e^{\mathrm{t}}w_1 + e^{\mathrm{t}}w_2 = 0$. Now $Tw_1 + T^{\mathrm{t}}w_2 = \gamma w_1$ and $(T^{\mathrm{t}} + I)w_1 + Tw_2 = \gamma w_2$, so that $Jw_1 + (J - I)\,w_2 = \gamma\,(w_1 + w_2)$. But $J\,(w_1 + w_2) = 0$ and Re $\gamma = -1/2$, so we find that $\gamma w_1 = \overline{\gamma} w_2$. In particular, both w_1 and w_2 are orthogonal to e. $Tw_1 + (J - I - T)\,w_2 = \gamma w_1$ implies that w_1 is an eigenvector of T, necessarily corresponding to an eigenvalue λ of T with Re $\lambda = -1/2$. Writing $w_1 = \sqrt{\overline{\overline{\lambda}}}u$ (where $Tu = \lambda u$), we find that $w_2 = xu$ for some scalar x. Using the eigenvalue–eigenvector relation we deduce that $x^2 = -\lambda$, and the results follows. $\square$

Corollary 4.3. *Suppose T is a tournament matrix of order n. Then* $2 \dim W_T^{\perp} = \dim W_{M_T}^{\perp}$, $2 \dim W_T = \dim W_{M_T}$ *and*

$$W_{M_T} = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : u, v \in W_T \right\}.$$

Proof. The fact that $2\dim W_T^{\perp} = \dim W_{M_T}^{\perp}$ follows from Theorem 4.2. We then deduce that $2\dim W_T = \dim W_{M_T}$ and

$$W_{M_T} = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : u, v \in W_T \right\}. \qquad \square$$

5. Determinants of tournament matrices in $\mathscr{M}_n$

We now investigate determinants for matrices in $\mathscr{M}_n$.

Theorem 5.1. *Let T be a tournament matrix of order n. Then*

$$\det(M_T) = (-1)^{n-1}(2n-1)\det\left(T - \frac{n-1}{2n-1}J\right). \tag{5.1}$$

Proof. Note that

$$\begin{bmatrix} I & I \\ O & I \end{bmatrix} M_T \begin{bmatrix} I & O \\ I & I \end{bmatrix} = \begin{bmatrix} 2J-I & J-I \\ J & T \end{bmatrix}.$$

Taking determinants and using the Schur complement of the last matrix above with respect to $2J-I$ yields $\det(M_T) = \det(2J-I)\det[T - J(2J-I)^{-1}(J-I)]$. Observe that

$$J(2J-I)^{-1}(J-I) = \frac{1}{2n-1}J(J-I) = \frac{n-1}{2n-1}J.$$

Eq. (5.1) now follows upon observing that $\det(2J-I) = (-1)^{n-1}(2n-1)$. $\square$

Corollary 5.2. *If T is an $n \times n$ tournament matrix with $n > 1$. Then*

$$\det M_T = (-1)(n-1)\det(T+I) + (-1)^{n-1}n\det(T).$$

Further $\det M_T < 0$ *and hence* $|\det M_T| = (n-1)\det(T+I) + (-1)^n n \det T$.

Proof. From Theorem 5.1,

$$\det M_T = (-1)^{n-1}(2n-1)\det\left(T - \left(\frac{n-1}{2n-1}\right)J\right).$$

Note that

$$\det(T - xJ) = \det\begin{bmatrix} t_{11}-x & t_{12}-x & \cdots & t_{1n}-x \\ t_{21}-x & t_{22}-x & \cdots & t_{2n}-x \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1}-x & t_{n2}-x & \cdots & t_{nn}-x \end{bmatrix}$$

$$= \det\begin{bmatrix} t_{11}-x & t_{12}-x & \cdots & t_{1n}-x \\ t_{21}-t_{11} & t_{22}-t_{12} & \cdots & t_{2n}-t_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{n1}-t_{11} & t_{n2}-t_{12} & \cdots & t_{nn}-t_{1n} \end{bmatrix}.$$

Thus $\det(T - xJ)$ is a linear function of x, say $\det(T - xJ) = a + bx$. Letting $x = 0$ yields $a = \det(T)$, while letting $x = 1$ yields $a + b = \det(T - J) = (-1)^n \det(T + I)$. Solving for b, substituting in $x = (n-1)/(2n-1)$, and simplifying now yields $\det M_T = (-1)(n-1)\det(T+I) + (-1)^{n-1} n \det T$.

Lastly we claim that $(n-1)\det(T+I) > |n\det(T)|$, so that in fact $|\det M_T| = (n-1)\det(T+I) + (-1)^n n \det T$. To see the claim, let the spectral radius of T be ρ, and suppose that its other eigenvalues are $x_j + \mathrm{i}y_j$, $1 \leqslant j \leqslant n-1$. Note that since $x_j \geqslant -1/2$ for all j, any real eigenvalue of $T + I$ is positive, so that $\det(T+I) > 0$.

In order to prove strict inequality we first assume $\rho = 0$, then clearly, $(n-1)\det(T+I) > |n\det(T)|$.

If $\rho > 0$, since $x_j \geqslant -1/2$ for all j and $\rho \leqslant (n-1)/2 < n-1$, we have

$$\begin{aligned}
\det(T+I) &= (\rho+1)\prod_{j=1}^{n-1}|x_j + \mathrm{i}y_j + 1| \\
&= (\rho+1)\prod_{j=1}^{n-1}\sqrt{x_j^2 + y_j^2 + 2x_j + 1} \\
&\geqslant (\rho+1)\prod_{j=1}^{n-1}\sqrt{x_j^2 + y_j^2} \\
&= \frac{\rho+1}{\rho}|\det(T)| > \frac{n}{n-1}|\det(T)|,
\end{aligned}$$

and the proof is complete. □

The next result gives a general upper bound on $|\det M_T|$ for $M_T \in \mathscr{M}_n$.

Theorem 5.3. *Let T be a tournament matrix of order n. Let $s = Te$ be the score vector for T. Then*

$$|\det(M_T)| \leqslant \frac{1}{(2n-1)^{n-1}}\prod_{i=1}^{n}[(n-1)^2 + n^2(n-1) - (2n-1)s_i]^{1/2} \tag{5.2}$$

and

$$|\det(M_T)| \leqslant \frac{1}{(2n-1)^{n-1}}\prod_{i=1}^{n}[n(n-1)^2 + (2n-1)s_i]^{1/2}. \tag{5.3}$$

Proof. From Theorem 5.1,

$$|\det(M_T)| = (2n-1)\left|\det\left(T - \frac{n-1}{2n-1}J\right)\right|.$$

Observe that in the ith row of $T-((n-1)/(2n-1))J$, there are s_i entries equal to $n/(2n-1)$ and $n-s_i$ entries equal to $(-n+1)/(2n-1)$. By Hadamard's inequality we find that

$$\left|\det\left(T-\frac{n-1}{2n-1}J\right)\right| \leqslant \prod_{i=1}^{n}\left(s_i\frac{n^2}{(2n-1)^2}+(n-s_i)\frac{(n-1)^2}{(2n-1)^2}\right)^{1/2}$$
$$=\frac{1}{(2n-1)^n}\prod_{i=1}^{n}[n(n-1)^2+(2n-1)s_i]^{1/2},$$

which yields (5.3). Note that

$$\det\left(T-\frac{n-1}{2n-1}J\right)=\det\left(T^{\mathrm{t}}-\frac{n-1}{2n-1}J\right) \quad \text{and} \quad T^{\mathrm{t}}e=(n-1)e-s,$$

so arguing as above readily establishes (5.2) . □

Corollary 5.4. *Let T be a tournament matrix of order n. Then*

$$|\det(M_T)| \leqslant \begin{cases} \frac{1}{(2n-1)^{n-1}}[(n-1)(n^2-\frac{1}{2})]^{n/2} & \text{if } n \text{ is odd},\\ \frac{1}{(2n-1)^{n-1}}[n^6-2n^5+2n^3-\frac{7}{2}n^2+\frac{n}{2}]^{n/4} & \text{if } n \text{ is even.}\end{cases}$$

Proof. Let s be the score vector of a tournament matrix of order n and consider the quantity $f(s)=\prod_{i=1}^{n}\left[n(n-1)^2+(2n-1)s_i\right]$. We claim that if s has two entries which differ by 2 or more, then there is another score vector $\tilde{s}$ of an $n\times n$ tournament matrix such that $f(s)<f(\tilde{s})$. To see the claim, suppose that we have $s_i \leqslant s_j+2$. Let $\tilde{s}$ be the vector with $\tilde{s_l}=s_l$ for $l\neq i, j$, $\tilde{s_i}=s_i+1$, and $\tilde{s_j}=s_j-1$. It follows from Landau's theorem [13] that $\tilde{s}$ is the score vector of some tournament matrix of order n. Further, since

$$\begin{aligned}&[n(n-1)^2+(2n-1)\tilde{s}_i][n(n-1)^2+(2n-1)\tilde{s}_j]\\ &\quad -[n(n-1)^2+(2n-1)s_i][n(n-1)^2+(2n-1)s_j]\\ &\quad =(2n-1)^2(s_j-s_i-1)>0,\end{aligned}$$

it follows that $f(\tilde{s})>f(s)$. Thus we see that $f(s)$ is maximized over the class of score vectors of order n by a vector $\overline{s}$ for which any two entries differ by at most 1. Thus for n odd each entry in $\overline{s}$ must be $(n-1)/2$, while for n even, half of the entries of $\overline{s}$ are $n/2$ and the rest are $(n-2)/2$. The upper bounds on $|\det M_T|$ are now established by computing $f(\overline{s})$ for the cases that n is odd or even. □

Remark 1. By examining the equality conditions for the Hadamard inequality it can be shown that the inequalities in Theorem 5.3 and Corollary 5.4 are strict.

The next few results deal with the determinant of a matrix in $\mathscr{M}_n$ arising from a regular tournament matrix T. Using part (ii) of Theorem 4.1 and setting $z = 0$ in the expression for $\det(zI - M_T)$ yields the following lemma.

Lemma 5.5. *Let T be a regular tournament matrix of odd order $n > 1$. Then* $\det(M_T) = -\det(T)$.

Theorem 5.6. *Let T be a regular tournament matrix of odd order $n > 1$. Then* $|\det(M_T)| \geqslant |\det(M_C)| = (n-1)/2$, *where C is the circulant tournament matrix of order n whose first row is*

$$[0, \overbrace{1, \ldots, 1}^{\frac{n-1}{2}}, \overbrace{0, \ldots, 0}^{\frac{n-1}{2}}].$$

Proof. Note that $|\det(M_T)| = |\det(T)|$, and since T is a regular tournament of odd order n, its row and column sums are $(n-1)/2$. Adding the last $n-1$ columns of T to the first column of T, we have that

$$|\det(M_T)| = \left|\det\begin{bmatrix} \frac{n-1}{2} & t_{12} & \cdots & t_{1n} \\ \frac{n-1}{2} & t_{22} & \ddots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n-1}{2} & t_{n2} & \cdots & t_{nn} \end{bmatrix}\right|$$

$$= \frac{n-1}{2}\left|\det\begin{bmatrix} 1 & t_{12} & \cdots & t_{1n} \\ 1 & t_{22} & \ddots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_{n2} & \cdots & t_{nn} \end{bmatrix}\right|$$

$$= \left(\frac{n-1}{2}\right)|q| \geqslant \left(\frac{n-1}{2}\right)$$

since $q \neq 0$ is an integer.

Using Lemma 5.5, $|\det(M_C)| = |\det(C)|$. Let P be the circulant permutation matrix whose first row is $[0\ 1\ 0\ \cdots\ 0]$. Then $C = P + P^2 + \cdots + P^{((n-1)/2)}$. Observe that $(I + P^{(n-1)/2})C = J - I$. Since $(I + P^{(n-1)/2})$ is permutationally similar to $I + P$ it has determinant 2. Thus $\det[(I + P^{(n-1)/2})C] = 2\det C = \det[J - I] = (-1)^{n-1}(n-1)$. The result now follows. $\square$

Conjecture 5.7. We conjecture that for any matrix M_T in the class $\mathscr{M}_n$, $|\det M_T| \geqslant \left\lfloor \frac{n}{2} \right\rfloor$.

Observe that for odd n, the matrix M_C of Theorem 5.6 provides an example for which equality holds in this conjectured lower bound; for n even, it can be shown that

$$\left|\det\begin{bmatrix} B_n & B_n^{\mathrm{t}} \\ B_n^{\mathrm{t}}+I & B_n \end{bmatrix}\right| = \left\lfloor \frac{n}{2} \right\rfloor.$$

A tournament matrix H of order n is a Hadamard tournament matrix if it satisfies the equation $HH^{\mathrm{t}} = ((n+1)/4)I + ((n-3)/4)J$ (necessarily $n \equiv 3 \bmod 4$). The existence of such matrices is a difficult unsolved problem since Hadamard tournaments of order n are coexistent with skew Hadamard matrices of order $n+1$. Using the fact that the determinant of the $n \times n$ matrix $aI + bJ$ is given by $(a+nb)\,a^{n-1}$, we obtain the following lemma.

Lemma 5.8. *If H is a Hadamard tournament matrix then*

$$\det H = \frac{(n-1)\,(n+1)^{(n-1)/2}}{2^n}.$$

Proof. Note that

$$\begin{aligned}(\det H)^2 = \det(HH^{\mathrm{t}}) &= \det\left(\frac{(n+1)}{4}I + \frac{(n-3)}{4}J\right) \\ &= \left[\frac{(n+1)}{4} + n\left(\frac{n-3}{4}\right)\right]\left(\frac{n+1}{4}\right)^{n-1} = \frac{(n-1)^2(n+1)^{n-1}}{2^{2n}}.\end{aligned}$$

The result follows. □

Theorem 5.9. *Suppose that T is an $n \times n$ regular tournament matrix with $n \geqslant 3$. Then*

$$|\det M_T| \leqslant \frac{(n-1)\,(n+1)^{(n-1)/2}}{2^n},$$

with equality if and only if T is a Hadamard tournament matrix.

Proof. By Lemma 5.5, $\det M_T = -1 \det T$. Using [19, Theorem 5.2, Chapter 8],

$$|\det M_T| = |\det T| \leqslant \frac{n-1}{2}\left(\frac{n-1}{2} - \frac{n-3}{4}\right)^{(n-1)/2} = \frac{(n-1)\,(n+1)^{(n-1)/2}}{2^n}.$$

Again appealing to [19, Theorem 5.2], equality holds if and only if T is an incidence matrix of a design, and hence T is a Hadamard tournament matrix. □

Remark 2. Let T be a tournament of order $n > 1$. Then since M_T is a nonsingular $(0, 1)$-matrix, $|\det(M_T)| \geqslant 1$.

The next two results deal with matrices in $\mathscr{M}_n$ whose associated T is singular.

Theorem 5.10. *Suppose that T is a singular $n \times n$ tournament matrix. Then $|\det M_T| \geqslant n - 1$, with equality if and only if M_T is permutationally similar to the Brualdi–Li matrix.*

Proof. From Corollary 5.2, $|\det M_T| = (n-1)\det(T+I)$. As argued in that Corollary, $T + I$ is nonsingular with positive determinant. Further, since $T + I$ is an integral matrix, we see that $\det(T + I) \geqslant 1$. This gives us the inequality on $|\det M_T|$.

We deal with the case of equality by proving that if $\det(T + I) = 1$, then T is transitive (the converse being obvious) and M_T is permutationally similar to the Brualdi–Li matrix. Suppose that T is singular but not transitive with eigenvalues ρ (the spectral radius), 0 of multiplicity k, and other nonzero eigenvalues $x_j + \mathrm{i}y_j$, $1 \leqslant i, j \leqslant n - k - 1$. Then

$$\begin{aligned}\det(T+I) &= (\rho+1)\left(\prod_{j=1}^{n-k-1}\left(\left(x_j+1\right)^2+y_j^2\right)\right)^{1/2}\\ &= (\rho+1)\left(\prod_{j=1}^{n-k-1}\left(x_j^2+2x_j+1+y_j^2\right)\right)^{1/2}\\ &> \rho\left(\prod_{j=1}^{n-k-1}\left(x_j^2+y_j^2\right)\right)^{1/2}.\end{aligned}$$

But this last term has the same absolute value as that of the coefficient of λ^k in the characteristic polynomial of T. In particular,

$$\rho\left(\prod_{j=1}^{n-k-1}\left(x_j^2+y_j^2\right)\right)^{1/2}$$

is a positive integer, so it is at least 1. Hence $\det(T + I) > 1$ if T is not transitive. $\square$

Theorem 5.11. *Suppose that T is a singular $n \times n$ tournament matrix. Then*

$$|\det M_T| \leqslant \frac{(n-1)\,n^{n/2}}{2^{n-1}},$$

with equality holding if and only if the eigenvalues of T are $(n-2)/2$, 0 *and* $-1/2 \pm (\mathrm{i}\sqrt{n-1})/2$, *the first two having multiplicity* 1, *and the last two each having multiplicity* $(n-2)/2$.

Proof. From Corollary 5.2, $|\det M_T| = (n-1)\det(T+I)$. Let the eigenvalues of T be 0, ρ (the spectral radius), and $x_j + \mathrm{i}y_j$, $1 \leqslant j \leqslant n-2$. Since the trace of T and T^2 are both 0,

$$\rho = -\sum_{j=1}^{n-2} x_j \quad \text{and} \quad \sum_{j=1}^{n-2} y_j^2 = \rho^2 + \sum_{j=1}^{n-2} x_j^2 = \left(\sum_{j=1}^{n-2} x_j\right)^2 + \sum_{j=1}^{n-2} x_j^2. \tag{5.4}$$

Using the fact that each $x_j \geqslant -1/2$, we have that $\rho \leqslant (n-2)/2$. This and the arithmetic–geometric mean inequality imply that

$$\begin{aligned} \det(T+I) &= (\rho+1)\prod_{j=1}^{n-2}\sqrt{1+2x_j+x_j^2+y_j^2} \\ &\leqslant \frac{n}{2}\left[\frac{1}{n-2}\sum_{j=1}^{n-2}(1+2x_j+x_j^2+y_j^2)\right]^{(n-2)/2}. \end{aligned} \tag{5.5}$$

If $\rho = 0$, we're done. Otherwise, $\rho \geqslant 1$. Observe that

$$\begin{aligned} &\sum_{j=1}^{n-2}(1+2x_j+x_j^2+y_j^2) \\ &\quad = n-2+2\sum_{j=1}^{n-2}x_j + \sum_{j=1}^{n-2}x_j^2 + \sum_{j=1}^{n-2}y_j^2 \\ &\quad = n-2+2\sum_{j=1}^{n-2}x_j + 2\sum_{j=1}^{n-2}x_j^2 + \left(\sum_{j=1}^{n-2}x_j\right)^2 \\ &\quad = \frac{n(n-2)}{4} + \sum_{j=1}^{n-2}(x_j+1/2)\left(2x_j + \sum_{k=1}^{n-2}x_k - \frac{(n-4)}{2}\right) \\ &\quad = \frac{n(n-2)}{4} + \sum_{j=1}^{n-2}(x_j+1/2)\left(2x_j - \rho - \frac{(n-4)}{2}\right). \end{aligned} \tag{5.6}$$

Now

$$0 = \operatorname{trace}(T) = \rho + x_j + \sum_{k\neq j} x_k \geqslant 2x_j - \frac{n-3}{2}$$

and $\rho \geqslant 1$. Hence, $2x_j - \rho - (n-3)/2 < -1$ for each j, and by (5.6) we conclude that

$$\sum_{j=1}^{n-2}(1 + 2x_j + x_j^2 + y_j^2) \leqslant \frac{n(n-2)}{4}.$$

The desired upper bound on $|\det M_T|$ now readily follows from (5.5).

Suppose that

$$|\det M_T| = \frac{(n-1)\,n^{n/2}}{2^{n-1}}.$$

From the argument above

$$\det(T+I) = \frac{n^{n/2}}{2^{n-1}},$$

$\rho = (n-2)/2$, each x_j is $-1/2$, and (from the case of equality for the arithmetic and geometric means) all the y_j^2's are the same. The expressions for the eigenvalues of T now follow. Conversely, if T has only the eigenvalues $(n-2)/2$, $-1/2 \pm (\mathrm{i}\sqrt{n-1})/2$ and 0 with the specified multiplicities, then the equality holds. □

Remark 3. From [12, Theorem 4], T has eigenvalues $(n-2)/2$, 0, and $-1/2 \pm (\mathrm{i}\sqrt{n-1})/2$ (the last pair with multiplicity $(n-2)/2$) if and only if $J - 2T$ is a skew–Hadamard matrix.

Conjecture 5.12. For any matrix M_T of order $2n$ in the class $\mathscr{M}_n$,

$$|\det M_T| \leqslant \frac{(n-1)\,n^{n/2}}{2^{n-1}}.$$

Remark 4. Observe that both the conjectured bound and the upper bound of Corollary 5.4 are asymptotic to $n^{n/2+1}/2^{n-1}$ for large n.

Lemma 5.13. *If T is an $n \times n$ tournament matrix, then*

$$\det(T+I) \leqslant \frac{(n+1)^{(n+1)/2}}{2^n}.$$

Proof. This follows from [19, Theorem 5.2]. □

Our last result confirms Conjecture 5.12 for the case that the associated T is reducible.

Theorem 5.14. *Suppose that T is an $n \times n$ reducible tournament matrix. Then*

$$|\det M_T| \leqslant \frac{(n-1)\,n^{n/2}}{2^{n-1}}.$$

Proof. Without loss of generality, write

$$T = \begin{bmatrix} T_1 & 0 \\ J & T_2 \end{bmatrix},$$

where T_1 is $k \times k$ and where T_2 is $(n-k) \times (n-k)$. If $1 \leqslant k \leqslant 2$ or $(n-2) \leqslant k \leqslant (n-1)$, then T is singular, and we are done by Theorem 5.11, so henceforth we take $3 \leqslant k \leqslant n-3$.

By an exhaustive search for the cases $n = 3, 4, 5, 6, 7$, and 8 (using the technique of [17] for generating these tournaments) the upper bound holds. Thus we can assume that $n \geqslant 9$.

From the proof of Corollary 5.2, it follows that $|\det M_T| \leqslant 2\,(n-1)\det(T+I)$, so we need only show that $\det(T+I) \leqslant n^{n/2}/2^n$. Observe that by Lemma 5.13,

$$\begin{aligned}\det(T+I) &= \det(T_1+I)\det(T_2+I) \\ &\leqslant \frac{(k+1)^{(k+1)/2}}{2^k}\,\frac{(n-k+1)^{(n-k+1)/2}}{2^{n-k+1}} \\ &= \frac{(k+1)^{(k+1)/2}\,(n-k+1)^{(n-k+1)/2}}{2^n}.\end{aligned}$$

It is easily seen that $(k+1)^{(k+1)/2}\,(n-k+1)^{(n-k+1)/2}$ is maximized for $3 \leqslant k \leqslant n-3$ at $k = 3$ or $n-3$, with a maximum value of $16\,(n-2)^{(n-2)/2}$. Since $n \geqslant 9$, $16\,(n-2)^{(n-2)/2} \leqslant n^{n/2}$, and so the result follows. $\square$

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